Kaluza-Klein Theories

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Abstract

In this short introduction to Kaluza-Klein (KK) theories, we emphasize on the possibility to obtain gauge fields from pure Einstein gravity in higher dimensions, a possibility introduced by Nordström (1914), Kaluza (1919), and Klein (1926).

The original idea to introduce one extra space dimension to obtain electromagnetism and gravity in four dimensions offers a way to unify the two known forces of the 19th century and give them a common, geometrical origin. This idea is then taken further to show that non-Abelian gauge fields can be obtained, corresponding to the weak and strong forces, by introducing more spatial dimensions.

A modern kind of Kaluza-Klein theory is represented by the theory of a universal extra dimension (UED), which can be seen as a generalization of the standard model of particle physics to five dimensions. It is shown that the four dimensional theory, as we observe it today, can be obtained, but with an extended particle spectrum that should be detectable at higher energies, depending on the size of the extra dimension.

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1 Introduction

The idea to unify concepts in physics is old. Already in the 19th century electricity and magnetism were united in what culminated in Maxwell's theory of electromagnetism. Due to the relativistic invariance of Maxwell's equations it was realized that the unification of electricity and magnetism was intimately connected with a unification of our three-dimensional space and the parameter describing time, yielding a continuous four-dimensional space-time. Just as space and time, being lower-dimensional objects in themselves, joined together in a higher-dimensional framework, the three-dimensional objects describing electricity and magnetism, $\mathbf{E}$ and $\mathbf{B}$, joined together as components of a single six-component anti-symmetric tensor $F_{\mu\nu}$ and the corresponding potentials ϕ and $\mathbf{A}$ unified in the four-vector A_μ.

When Einstein's gravitational theory of general relativity became available, efforts were made to unite gravity with the electromagnetic force. The hope to see this unification manifested in higher dimension was initiated by Kaluza [1] and Klein [2], in what today is known as Kaluza-Klein theories.

2 The five-dimensional theory

The results obtained in these sections follows that of Refs. [3, 4, 6]. The convention used is that the space-time metric is expanded around a ground state where the four-dimensional part is given by

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1).$$

The five-dimensional Kaluza-Klein theory starts from a generalization of the free, four-dimensional, Einstein action of general relativity to five dimensions (one temporal and four spatial):

$$\hat{I} = -\frac{1}{16\pi\hat{G}}\int d^5\hat{x}\sqrt{-\hat{g}}\hat{R}, \tag{1}$$

where the hat indicates five-dimensional quantities; $\hat{G}$ is the gravitational constant in five dimensions, $\hat{R}$ is the scalar curvature formed from the five-dimensional metric $\hat{g}_{MN}$ and $\hat{g} = \det\hat{g}_{MN}$. Capital indices M, N are used to denote components of the full-dimensional quantities and Greek indices μ, ν to denote the ordinary, four-dimensional, components. The fifth dimension x^4 will be denoted by y. A general expression for the metric serving our purpose is given by

$$\hat{g}_{MN} = \begin{pmatrix} g_{\mu\nu} - \xi^2 A_\mu A_\nu \phi & \xi A_\mu \phi \\ \xi A_\nu \phi & -\phi \end{pmatrix}, \tag{2}$$

where ϕ is a scalar quantity and ξ is a factor that later will be used to normalize A_μ. In the most general case the ten component, four-dimensional, metric $g_{\mu\nu}$, the four-vector A_μ, and the scalar ϕ can depend on all five space-time coordinates. To extract the Abelian gauge field, i.e., electromagnetism and gravity from Eq. (2), it is, however, sufficient to remove all y dependence from Eq. (2) and also put ϕ equal to a constant. This was Kaluza's original approach, but he failed to give an adequate motivation to this restriction.

The special form of the five-dimensional metric (2) allows for coordinate transformations in the fifth dimension to be interpreted as gauge transformations. Consider a transformation of the fifth dimension given by

$$y \to y' = y + \xi\epsilon(x). \tag{3}$$

For a general coordinate transformation

$$\hat{g}_{AB} = \hat{g}'_{A'B'}\frac{\partial \hat{x}'^{A'}}{\partial \hat{x}^A}\frac{\partial \hat{x}'^{B'}}{\partial \hat{x}^B} \tag{4}$$

the invariance of the metric is assured if A_μ transforms as

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \epsilon(x). \tag{5}$$

This is recognized as an Abelian gauge transformation of A_μ, and thus, the extra dimension displays a symmetry in coordinate transformation equal to the one that classically results in the electromagnetic four-vector. It would, however, be wrong to call A_μ a photon just because of this symmetry, after all it has to obey Maxwell's equations. This was the remarkable thing about Kaluza's approach. Upon splitting the five-dimensional Einstein equations in vacuum $\hat{R}_{MN} = 0$[1], where $\hat{R}_{MN}$ is the five-dimensional Ricci tensor derived from the metric (2), into its lower-dimensional components he arrived at the equations

$$\begin{aligned} R_{\mu\nu} &= 0, && (6)\\ \partial^\mu F_{\mu\nu} &= 0, && (7)\\ \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} &= 0, && (8) \end{aligned}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. These are precisely the four-dimensional Einstein and Maxwell equations in vacuum. Thus, Maxwell's theory of electromagnetism can be considered as an inevitable consequence of general relativity, if there is an extra dimension.

However, some obvious questions appears if this is to be a physical theory. For instance, there is no obvious reason why the five-dimensional metric should be independent of y or, a much larger problem, where is the fifth dimension?

3 A compact fifth dimension

The next step in higher-dimensional unification was taken by Klein in 1926, who continued Kaluza's work by imposing that the fifth dimension was compactified on some scale $\tilde{R}$, i.e., the fifth dimension forms a circle with radius $\tilde{R}$ (we can think of the five-dimensional manifold as $M = M_4 \times S_1$ with a four-dimensional space-time manifold M_4 and S_1 being the circle). This enables any field quantity $F(x, y)$ to be expanded in a Fourier expansion:

$$F(x,y) = \sum_{n=-\infty}^{\infty} F_n(x) \exp\left(iny/\tilde{R}\right), \quad 0 \le y \le 2\pi\tilde{R}. \tag{9}$$

If the field quantities $g_{\mu\nu}$, A_μ, and ϕ are expanded in this way and only the zero mode of the expansions are considered, one naturally obtains fields that only depend on x, such as Kaluza's original fields. Shortly, we will motivate that the higher modes correspond to mass eigenstates with masses on the Planck scale, thus motivating why only the zero mode harmonics need to be considered in the effective, low-energy four-dimensional theory.

To obtain the effective four-dimensional theory we insert the metric anzats (2) in the free Einstein action (1), compute the five-dimensional curvature scalar $\hat{R}$ for the zero mode fields (keeping ϕ constant equal to unity), and integrate over the extra dimension. The result is [3]:

$$I = -\frac{2\pi\tilde{R}}{16\pi\hat{G}}\int d^4x\sqrt{-g}\left(R + \frac{\xi^2\tilde{R}^2}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{10}$$

[1] In vacuum, the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ reduces to $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$ since the energy-momentum tensor $T_{\mu\nu}$ is zero. Multiplying this by the inverse metric $g^{\mu\nu}$ turns the vacuum equations into $R_{\mu\nu}$=0, which, in five dimensions, is generalized to $\hat{R}_{MN} = 0$.

where R is the four-dimensional curvature scalar. From this the gravitational constant in four dimensions G can be identified as

$$G = \frac{\hat{G}}{2\pi\tilde{R}}. \tag{11}$$

If we further make the normalization

$$\xi^2 = 16\pi G, \tag{12}$$

Eq. (10) turns into

$$I = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}, \tag{13}$$

which is the four-dimensional Einstein-Maxwell action. One can note that if the extra dimension had been timelike, there would have been a wrong relative sign between the gravity and electromagnetic part of the action.

If we allow for the scalar field ϕ in the metric (2) to depend on the coordinates x, then, after compactification, there will also be a contribution to the four-dimensional action (13) with density proportional to $\partial_\mu\sigma\partial^\mu\sigma$, where σ is proportional to $\ln\phi$.

4 Mass eigenstates and charge quantization

The idea of introducing the extra dimension as compactified on a circle allows for an expansion of any field in Fourier modes as in Eq. (9). Consider the five-dimensional wave equation of a scalar field $\phi(x, y)$, i.e., a massless scalar field in five dimensions, expanded in Fourier modes:

$$(\partial_t^2 - \nabla_x^2 - \partial_y^2)\sum_n \phi_n(x)\exp\left(iny/\tilde{R}\right) = 0. \tag{14}$$

Taking the derivative of the y-dependent part yields for the Fourier modes:

$$(\partial_t^2 - \nabla_x^2 + m_n^2)\phi_n(x) = 0, \tag{15}$$

where

$$m_n^2 = \left(\frac{n}{\tilde{R}}\right)^2. \tag{16}$$

Thus, the fields $\phi_n(x)$ are massive eigenstates in four dimensions with mass m_n. To find the scale of these massive fields, we note that the coordinate transformation (3) leads to another feature of the fields ϕ_n. The transformation $y \to y' = y + \xi\epsilon(x)$ yields for ϕ_n:

$$\phi_n(x) \to \phi'_n(x) = \exp\left(in\xi\epsilon(x)/\tilde{R}\right)\phi_n(x). \tag{17}$$

This transformation leaves the action resulting in the Klein-Gordon equation (15) invariant, and thus, since the fields A_μ transform according to the gauge transformation (5), an infinitesimal local coordinate transformation $\epsilon(x)$ of the fifth dimension associates, trough Nöther's theorem, a charge q_n to the fields $\phi_n(x)$, with

$$q_n = -\frac{n\xi}{\tilde{R}} = -n\frac{4\sqrt{\pi G}}{\tilde{R}}. \tag{18}$$

One remarkable feature of the theory is that the charge is quantized in units of $4\sqrt{\pi G}/\tilde{R}$. If we dare to identify the charge q_1 with the elementary charge e, the scale of the fifth dimension is given by

$$\tilde{R} = \frac{4\sqrt{\pi G}}{e} \approx 10^{-32}\,\mathrm{m}, \tag{19}$$

which is indeed close to the Planck scale of about 10^{-34} m. The fact that the extra dimension is compactified on such a small scale is indeed a triumph for the theory as to motivate why the extra dimension never has been observed. It is, however, apparent that the five-dimensional theory obtained so far cannot describe electromagnetism. According to Eqs. (16) and (18), the lightest particle with a charge would acquire a mass close to the Planck mass. All known particles have, however, a very low mass on that scale.

5 Higher-dimensional Kaluza-Klein theory and non-Abelian gauge fields

As we have seen, it is possible to retain an effective four-dimensional theory of electromagnetism and gravity, starting from only gravity in five dimensions. The next step would be to try and unify gravity with the two other known forces of Nature, the weak and strong forces.

To do this we need the extra dimensions to form a compact manifold with isometries. An isometry of a manifold, with coordinates y^n, is a coordinate transformation $y^n \to y'^n$ such that the metric, h_{mn}, of the compact manifold is invariant, i.e.,

$$y^n \to y'^n : h'_{mn}(y'^n) = h_{mn}(y^n). \tag{20}$$

It can be shown that isometries form a group with generators t_a and structure constants C_{abc} with a general, infinitesimal coordinate transformation given by

$$y^n \to y'^n = y^n + \epsilon^a \xi^n_a(y^n), \tag{21}$$

where ξ^n_a are the Killing vectors with higher indices running over the N extra dimensions and lower indices running over the dimensions of the isometry group and ϵ^a is an infinitesimal parameter independent of y. The Killing vectors that are associated with the independent infinitesimal isometries obey the algebra:

$$\xi^m_b \partial_m \xi^n_c - \xi^m_c \partial_m \xi^n_b = -C_{abc} \xi^n_a. \tag{22}$$

From this it can be showed that the generators of the symmetry group obeys

$$[t_a, t_b] = i C_{abc} t_c. \tag{23}$$

The isometry groups of the manifolds S_1 (circle) and S_2 (sphere) are given by the groups $SO(2)$ and $SO(3)$, or $U(1)$ and $SU(2)$, respectively. To reproduce the standard model of particle physics, it would be necessary to chose a compact manifold with isometry group containing $SU(3) \times SU(2) \times U(1)$, which is the observed gauge group of the electroweak and strong interactions.

The non-Abelian gauge fields may arise from a generalization of the metric (2) in $4 + D$ dimensions given by

$$\hat{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x) - h_{mn}(y) B^m_\mu B^n_\nu & B^n_\mu \\ B^m_\nu & -h_{mn}(y) \end{pmatrix}, \tag{24}$$

where $x^A = (x, y) = (x^\mu, y^m)$ with $\mu = 0, ..., 3$, $m = 4, ..., D - 1$. We have thus added to the metric $D - 4$ four-vectors

$$B^n_\mu = \xi^n_a(y) A^a_\mu(x) \tag{25}$$

and it is the transformation of these vectors, under a x^n-dependent isometry of the same kind as Eq. (21), which will give the non-Abelian gauge fields. Under a coordinate transformation

$$y^n \to y^n + \xi^n_a(y) \epsilon_a(x), \tag{26}$$

the elements of the higher-dimensional metric transform as

$$A^a_\mu \to A^{a\,\prime}_\mu = A^a_\mu + \partial_\mu \epsilon^a(x) + C_{abc} \epsilon^b(x) A^c_\mu, \tag{27}$$

which can be recognized as the usual Yang-Mills gauge transformations, i.e., the A^a_μ's are non-Abelian gauge fields.

In a similar way as in five dimensions, the effective four-dimensional theory can be obtained by inserting the anzats (24) as the metric in the $4 + D$-dimensional Einstein gravity action

$$\hat{I} = -\frac{1}{16\pi \hat{G}} \int d^{4+D} \hat{x} \sqrt{-\hat{g}} \hat{R}. \tag{28}$$

Reducing $\hat{R}$ to a four-dimensional curvature scalar R will give some extra terms and after separating the integrals over four and D dimensions, the action (28) becomes [3]

$$I = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{4} \int d^4x \sqrt{-g} F^a_{\mu\nu} F^{a\mu\nu} \tag{29}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - C_{abc} A^b_\mu A^c_\nu. \tag{30}$$

In Eq. (29), the gravitational constant in four dimensions have been identified as

$$\frac{1}{G} = \frac{1}{\hat{G}} \int d^D y \sqrt{-h}, \tag{31}$$

where $h = \det h_{mn}(y)$ and the Killing vectors must obey

$$\frac{1}{16\pi\hat{G}} \int d^D y \sqrt{-h} \xi_a^m \xi_b^n h_{mn} = \delta_{ab}. \tag{32}$$

The action (29) is indeed the standard action for gravity and non-Abelian gauge fields in four dimensions. The metric (24) can be made more general by including x and y dependence in all fields. This will typically lead to more fields present in the effective four-dimensional Lagrangian.

The problem with the scenario above is, however, that it implies that our four-dimensional space cannot be flat. The Einstein equations in vacuum read $\hat{R}_{MN} = 0$ where the Ricci tensor is formed from the metric (24) but with all gauge fields set to zero. If four-space is to be flat $\hat{R}_{MN}$ must vanish for the space time indices and thus it must vanish for the higher dimensional indices as well. This can however not be the case if the higher dimensional space is curved, which is the case for manifolds with non-Abelian isometry groups.

6 Universal extra dimensions

The model of universal extra dimensions (UED) is basically a higher-dimensional generalization of the standard model of particle physics (here we will disregard the strong force). As a consequence, all the ordinary standard model particles are accompanied by a tower of massive states originating from the same kind of reasoning which led to Eq. (16). This is referred to as the Kaluza-Klein (KK) tower and its masses are given by:

$$M_n = \sqrt{M_{EW}^2 + \left(\frac{n}{\tilde{R}}\right)^2}, \tag{33}$$

where M_{EW} is the original electroweak mass. Instead of deriving the gauge fields from a special kind of higher-dimensional metric, trough compactification, we can obtain the gauge fields from a higher-dimensional Lagrangian:

$$\hat{\mathcal{L}} = -\frac{1}{4}\hat{F}_{MN}\hat{F}^{MN} - \frac{1}{4}\hat{F}_{MN}^{\alpha}\hat{F}^{\alpha MN}, \tag{34}$$

where $\hat{F}_{MN}$ and $\hat{F}_{MN}^{\alpha}$ are the five-dimensional $U(1)$ and $SU(2)$ field strengths, respectively. The M, N indices ranges from 0 to 4 and α over the dimensions of the symmetry group, 1 to 3 in the case of $SU(2)$. The four-dimensional theory is regained after inserting the appropriate expansions of the higher-dimensional fields, in analogy with Eq. (9), integrating over the extra dimension, and keeping only the zero modes from the expansions.

There is, however, a problem with compactification on a circle this time which originates from the fact that the fifth component of the five-dimensional vector field transforms as a scalar under four-dimensional Lorentz transformations. Hence, the zero mode contribution from the fifth component of the vector field will give rise to light scalar fields. Such fields are, however, not observed, and thus, they must be prevented from appearing in the effective four-dimensional theory. This can be done by compactifying on the *orbifold* S_1/Z_2 instead of on the circle S_1. This orbifold not only has a symmetry under $y \to y + 2\pi R$, but also a symmetry under $y \to -y$, so we have selected a point on the circle around which we have a mirror symmetry. The field can either transform odd or even around this point and depending on which there will either be a sine or cosine expansion of the fields, instead of a combination as in the S_1 compactification, cf. Eq. (9). By imposing an odd transformation on the fifth component of the vector fields, there will be no zero modes, and thus, none will appear in the 4D-theory.

In analogy with the electroweak theory, a complex $SU(2)$ doublet

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^2 + i\chi^1 \\ H - i\chi^3 \end{pmatrix}, \quad \chi_\pm = \frac{1}{\sqrt{2}}(\chi^1 \mp i\chi^2) \tag{35}$$

in the five-dimensional Lagrangian

$$\hat{\mathcal{L}}_{Higgs} = (D_M\Phi)^\dagger(D^M\Phi) - V(\Phi), \tag{36}$$

can be introduced to provide masses for the gauge fields by spontaneous symmetry breaking provided by the prototype 'mexican hat' potential:

$$V(\Phi) = -\mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2, \quad \lambda = \frac{\mu^2}{v^2} = \frac{m_H^2}{2v^2}, \tag{37}$$

where m_H is the mass and v the vacuum expectation value of the scalar Higgs field H. The covariant derivative in Eq. (36) is given by

$$D_N = \partial_N - i\hat{g}A_N^\alpha \frac{\sigma^\alpha}{2} - \frac{i}{2}\hat{g}_Y B_N, \tag{38}$$

where we have set the hyper charge to 1/2, $\hat{g}_Y$ and $\hat{g}$ are the higher-dimensional $U(1)$ and $SU(2)$ coupling constants, respectively, and σ^α are the three Pauli matrices. To arrive at the correct four-dimensional theory after integrating over the fifth dimension, the four- and five-dimensional coupling constants always have to be related as

$$g = \frac{1}{\sqrt{2\pi\tilde{R}}}\hat{g}. \tag{39}$$

Through mixing angels the gauge fields can be combined to give rise to the photon and the three vector bosons of weak interaction:

$$W_M^\pm = \frac{1}{\sqrt{2}}(A_M^1 \mp iA_M^2), \tag{40}$$

$$A_M = c_w B_M + s_w A_M^3, \tag{41}$$

$$Z_M = c_w A_M^3 - s_w B_M \tag{42}$$

with

$$s_w = \sin\theta_w = \frac{\hat{g}_Y}{\sqrt{\hat{g}^2 + \hat{g}_Y^2}}, \quad c_w = \cos\theta_w = \frac{\hat{g}}{\sqrt{\hat{g}^2 + \hat{g}_Y^2}}, \tag{43}$$

so that the elementary charge is given by $e = s_w g = c_w g_Y$. The full spectrum of particles in the effective four-dimensional theory will be given by the KK vector modes $A_\mu^{(n)}$, $Z_\mu^{(n)}$, and $W_\mu^{\pm(n)}$ together with four Goldstone bosons to generate their masses. In contrast to the standard model, there will also be four scalar fields present. By picking the orbifold transformation in an appropriate way we can, however, retain only the three Goldstone bosons, giving mass to the zero mode vector fields $A_\mu^{(0)}$, $Z_\mu^{(0)}$, and $W_\mu^{\pm(0)}$, and the zero mode scalar Higgs field $H^{(0)}$ of the observed standard model.

Since none of the particles introduced by UED ever have been observed, this sets a restriction on the mass scale of the KK excitations, and hence, on the size of the extra dimension. There is an interesting possibility that some of the KK excitations might be dark matter candidates, so-called WIMPs. One such would be the first excitation of the weak hyper charge gauge boson, $B^{(1)}$. Indeed, if the extra dimension would be on the TeV scale, the relic density of the $B^{(1)}$ from the Big Bang would be just right today. By studying cosmic rays from, for instance, the galactic center, one can look for signatures of $B^{(1)}$ annihilations which, if they were found, would support a theory with extra dimensions and help solving the dark matter problem. A study of this, and more on UED aspects on cosmology, can be found in Ref. [7].

7 Summary and outlook

In this short introduction to Kaluza-Klein theory, we have seen the possibility to obtain Abelian and non-Abelian gauge fields starting from Einstein gravity in higher dimensions. We have seen that if the extra dimensions are compactified on a small scale, it is possible to obtain many of the features of our observed, four-dimensional physics on small energy scales. All coupling constants of the effective theory can be seen to depend on the size of the extra dimensions, and in the UED theory, the extra dimensions are expected to reveal their presence in form of a large particle spectrum carrying high masses, inversely proportional to the size of the extra dimensions which is expected to be small.

There are, however, strong reasons to believe that our physical world cannot be derived from higher dimensions where there are no fields but gravity present. Besides the problem to obtain non-Abelian gauge fields, mentioned in section 5, there is a problem to obtain light, chiral fermions. One way of resolving both these problems may be to assume that some gauge fields are already present in the higher-dimensional theory and that the connection between these elementary gauge fields and gravitational fields will form a consistent theory. A promising gauge group to achieve this is given by $E_8 \times E_8$ or $SO(32)$, which is a supergravity limit of superstring theory. Although the pure Kaluza-Klein theory might not be the final answer as to unify all the forces of physics, the concept of extra spatial dimensions seams to play a very important role in modern, theoretical physics.

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